導来圏 et 導来函手 en Géométrie Algébrique

Peize Liu

22 August 2022

Contents

References

Expository notes:

- Jinghui Yang & Shuwei Wang, *Triangulated categories and derived categories* [YS]
- Schapira, *Categories and Homological Algebra.* [Scha]
- Bridgeland, *D^b (Intro).*
- Căldăraru, *Derived Categories of Sheaves: A Skimming.* [Căld]
- Calabrese, *On a Theorem of Beĭlinson.*
- Craw, *Quiver Representations in Toric Geometry.* [Craw]

Books:

- Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry.* [Huyb]
- Hartshorne, *Algebraic Geometry.* [HartsAG]
- Hartshorne, *Residues and Duality.* [HartsRD]
- 李文威，*代数学方法* II (未定稿). [李文威]
- Bocklandt, *A Gentle Introduction to Homological Mirror Symmetry.* [Bock]

Prerequisites (Oxford courses):

- B2.2 Commutative Algebra
- C2.2 Homological Algebra
- C2.6 Introduction to Schemes
- C3.4 Algebraic Geometry

I will take everything from those courses for granted.

Overview

Kontsevich's homological mirror symmetry is a conjecture on the derived equivalence of the A_∞-categories

$$\mathsf{D}^\pi\mathsf{Fuk}(X) \simeq \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X^\vee)$$

for a mirror pair $(X, X^\vee)$ of Calabi–Yau varieties. The left-hand side is the derived Fukaya category constructed from the symplectic geometry of X, known as the A-model, whereas the right-hand side is the bounded derived category of coherent sheaves on $X^\vee$, known as the B-model. These notes aim to fill in the gaps between undergraduate algebraic geometry and the essential backgrounds of understanding $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$ when X is a smooth projective variety.

Some topics and results in derived categories of sheaves to be covered:

- Some initial results, e.g. $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \cong \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathsf{QCoh}(X))$;
- Smoothness, perfect complexes;
- Serre duality, Serre functor, Grothendieck–Verdier duality;
- Ampleness, canonical bundle, Fano & Calabi–Yau varieties;
- Bondal–Orlov reconstruction theorem: For X, Y smooth projective variety with X Fano or anti-Fano, if $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \cong \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$, then $X \cong Y$;
- Fourier–Mukai transforms, Orlov's result on equivalences between derived categories;
- Beĭlinson's resolution, derived category of projective n-spaces $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(\mathbb{P}^n)$;
- $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(\mathbb{P}^1) \simeq \mathsf{D}^{\mathrm{b}}\,\mathsf{Rep}\,Q$ for the Kronecker quiver Q.

I will continue from the notes ([YS]) *Triangulated categories and derived categories* by Jinghui Yang & Shuwei Wang. **Warning.** Currently these notes grew out from a talk and was not self-contained in nature. In the future they may be extended to a more inclusive version, where I aim to present derived categories and localisations rigourously.

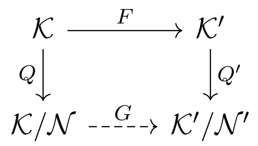
0 Derived Functors

This section mainly follows [李文威]. The relevant sections are 1.8, 1.11, 3.2, 4.6–4.9, 4.12.

Recall that from an Abelian category $\mathcal{A}$ we can build the **homotopy category** $\mathsf{K}(\mathcal{A})$ by taking quotient by chain maps homotopic to zero in the chain complex category $\mathsf{Ch}(\mathcal{A})$, and the **derived category** $\mathsf{D}(\mathcal{A})$ by (Verdier) localisation on the acyclic complexes in $\mathsf{K}(\mathcal{A})$. In particular, every quasi-isomorphism of chains in $\mathcal{A}$ becomes an isomorphism in $\mathsf{D}(\mathcal{A})$ (and $\mathsf{D}(\mathcal{A})$ is universal with respect to this property by construction). In general, $\mathsf{K}(\mathcal{A})$ and $\mathsf{D}(\mathcal{A})$ are not Abelian, but rather **triangulated categories**. For all the technical details we refer to the notes from the previous talk. If $\mathcal{A}$ has enough injectives, then $\mathsf{D}^+(\mathcal{A})$ is equivalent to $\mathcal{I}_{\mathcal{A}}$, the full subcategory of injective objects of $\mathcal{A}$.

There is a natural way to define derived functor under the viewpoint of derived categories. First we recall the classical definition. Suppose that $\mathcal{A}$ is an Abelian category with enough injectives. For $A \in \mathrm{Obj}(\mathcal{A})$, let $A \to I^\bullet$ be an injective resolution of A. Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor. Then the ***n*-th right derived functor** of F acting on X is given by $\mathsf{R}^n F(A) := \mathrm{H}^n(F(I^\bullet))$.

Let $\mathcal{K}$ and $\mathcal{K}'$ be triangulated categories, and $Q\colon \mathcal{K} \to \mathcal{K}/\mathcal{N}$ and $Q'\colon \mathcal{K}' \to \mathcal{K}'/\mathcal{N}'$ be Verdier localisations. Suppose that $F\colon \mathcal{K} \to \mathcal{K}'$ is a triangulated functor (i.e. preserving distinguished triangles). The naive idea is to seek for a functor G such that the following diagram commutes (and satisfies some universal properties):

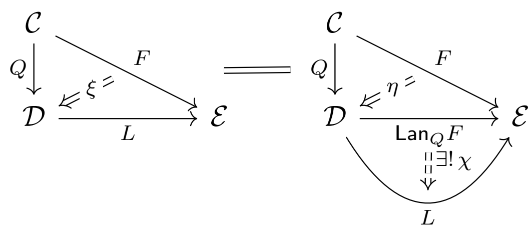


For this we need the Kan extension from category theory. Let's recap.

Definition 0.1. Consider functors $Q\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$. The **left Kan extension** of F by Q consists of the following data:

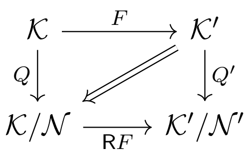
- A functor $\mathsf{Lan}_Q F\colon \mathcal{D} \to \mathcal{E}$;
- A natural transformation $\eta\colon F \Rightarrow \mathsf{Lan}_Q F \circ Q$;

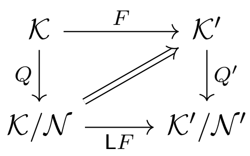
which satisfy the following universal property: for any functor $L\colon \mathcal{D} \to \mathcal{E}$ and natural transformation $\xi\colon F \Rightarrow L \circ Q$, there exists a unique $\chi\colon \mathsf{Lan}_Q F \Rightarrow L$ such that $\xi = (\chi \circ Q) \circ \eta$.



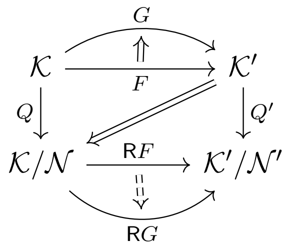
Considering left Kan extension in the opposite categories, we could define **right Kan extension**. The corresponding diagram is given by reversing all natural transformations in the above diagram.

Definition 0.2. Let $F\colon \mathcal{K} \to \mathcal{K}'$ as above. If the left (*resp.* right) Kan extension $\mathsf{Lan}_Q(Q' \circ F)$ (*resp.* $\mathsf{Ran}_Q(Q' \circ F)$) exists and is a triangulated functor, then it is called the right (*resp.* left) **derived functor** of F, denoted by $\mathsf{R}F$ (*resp.* $\mathsf{L}F$).

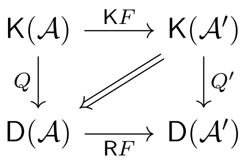


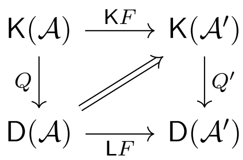


Remark. Suppose that $G\colon \mathcal{K} \to \mathcal{K}'$ is another triangulated functor with a natural transformation $\eta\colon F \Rightarrow G$. If the right derived functor $\mathsf{R}G$ exists, then there is a canonical natural transformation $\mathsf{R}F \Rightarrow \mathsf{R}G$ by the universal property of right Kan extension.



Then we focus on the derived categories. Note that an additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ between Abelian categories induces the homotopy functor $\mathsf{K}F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$[1] which is triangulated. Consider the Kan extensions:





Assuming existence, $\mathsf{R}F$ (*resp.* $\mathsf{L}F$) is called the right (*resp.* left) derived functor of F. Their uniqueness is ensured by the universal property. What about existence?

Definition 0.3. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be as above. Let $\mathcal{J}$ be a triangulated subcategory of $\mathsf{K}(\mathcal{A})$. We say that $\mathcal{J}$ is ***F*-injective** (*resp.* ***F*-projective**), if:

- Resolution: For $X \in \mathrm{Obj}(\mathsf{Ch}(\mathcal{A}))$ there exists $Y \in \mathrm{Obj}(\mathcal{J})$ and a quasi-isomorphism $X \to Y$ (*resp.* $Y \to X$).
- Preserving null system: $F(\mathrm{Obj}(\mathcal{N}(\mathcal{A}) \cap \mathcal{J})) \subseteq \mathrm{Obj}(\mathcal{N}(\mathcal{A}'))$

Note that here the null system $\mathcal{N}(\mathcal{A})$ is the acyclic complexes in $\mathsf{Ch}(\mathcal{A})$.

Remark. There is a similar notion for subcategories of $\mathcal{A}$. Let $\mathcal{I}$ be an additive full subcategory of $\mathcal{A}$. We say that $\mathcal{I}$ is of **type I** (*resp.* **type P**) relative to F, if:

- For any $X \in \mathrm{Obj}(\mathcal{A})$ there exists $Y \in \mathrm{Obj}(\mathcal{I})$ and a monomorphism $X \to Y$ (*resp.* epimorphism $Y \to X$);
- For any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if $X, Y \in \mathrm{Obj}(\mathcal{I})$ then $Z \in \mathrm{Obj}(\mathcal{I})$. (*resp.* If $Y, Z \in \mathrm{Obj}(\mathcal{I})$ then $X \in \mathrm{Obj}(\mathcal{I})$.) In this case $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is also exact.

This should be considered as the generalisation of injective objects in $\mathcal{A}$. Indeed the subcategory $\mathcal{I}_{\mathcal{A}}$ of injective objects of $\mathcal{A}$ is of type I relative to any additive functor F.

[1] The cases for K^+, K^-, and K^{b} are identical.

The terminology is taken from [李文威, 4.8.2]. In fact, this notion is what [Scha, 4.7.5] calls *F-injective*. The two definitions are closely related. If $\mathcal{I} \subseteq \mathcal{A}$ is of type I relative to F, then $\mathsf{K}(\mathcal{I}) \subseteq \mathsf{K}(\mathcal{A})$ is F-injective.

Proposition 0.4

Let $F\colon \mathcal{A} \to \mathcal{A}'$ be as above. Suppose that $\mathsf{K}(\mathcal{A})$ has an F-injective (*resp.* F-projective) subcategory. Then the right (*resp.* left) derived functor $\mathsf{R}F$ (*resp.* $\mathsf{L}F$) exists.

Proof. Let $\mathcal{I}$ be an F-injective subcategory of $\mathsf{K}(\mathcal{A})$. By Theorem 3.5 in [YS], there is an equivalence of category $\mathsf{D}(\mathcal{A}) \simeq \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I})$. Since $F(\mathrm{Obj}(\mathcal{N}(\mathcal{A}) \cap \mathcal{I})) \subseteq \mathrm{Obj}(\mathcal{N}(\mathcal{A}'))$, by the universal property of Verdier localisation there is a functor $F^\flat\colon \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I}) \to \mathsf{D}(\mathcal{A}')$. Take $\mathsf{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$ to be the functor such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{D}(\mathcal{A}) & \xrightarrow{\mathsf{R}F} & \mathsf{D}(\mathcal{A}') \\ {\scriptstyle i^{-1}}\uparrow\downarrow{\scriptstyle i} & \nearrow_{F^\flat} & \\ \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I}) & & \end{array}$$

Next we need to verify that $\mathsf{R}F$ is indeed the Kan extension. See [李文威, Prop 1.11.2, Prop 4.6.4]. ☐

Corollary 0.5

Suppose that $\mathcal{A}$ has enough injectives (*resp.* projectives). Then the right (*resp.* left) derived functor ${}^{+}\mathsf{R}F$ (*resp.* ${}^{+}\mathsf{L}F$) exists for any additive functor $F\colon \mathcal{A} \to \mathcal{A}'$.

Proof. Immediate by [YS, Prop 3.10]. ☐

Proposition 0.6

Suppose that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be a left exact additive functor. Then for $A \in \mathrm{Obj}(\mathcal{A})$. we have

$$\mathsf{R}^n F(A) = \mathrm{H}^n \circ \mathsf{R}F(QA),$$

where $QA \in \mathsf{D}^+(\mathcal{A})$ and $\mathrm{H}^n\colon \mathsf{D}^+(\mathcal{A}') \to \mathsf{Ab}$ is the n-th cohomology functor.

Proof. Take an injective resolution $A \to I^\bullet$. This gives rise to a quasi-isomorphism $A \to I$ in $\mathsf{K}^+(\mathcal{A})$, where I lies in the F-injective subcategory $\mathsf{K}^+(\mathcal{I}_\mathcal{A})$ of $\mathsf{K}^+(\mathcal{A})$. Now we have the isomorphisms

$$\mathsf{R}F(QA) \cong \mathsf{R}F(QI) \cong Q'\mathsf{K}^+F(I).$$

Applying H^n gives the result. ☐

Proposition 0.7. Long Exact Sequence

Suppose that $F\colon \mathcal{A} \to \mathcal{A}'$ has a right derived functor $\mathsf{R}F$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathsf{D}(\mathcal{A})$, there is a canonical long exact sequence:

$$\cdots \longrightarrow \mathsf{R}^{n-1}(Z) \longrightarrow \mathsf{R}^n F(X) \longrightarrow \mathsf{R}^n F(Y) \longrightarrow \mathsf{R}^n F(Z) \longrightarrow \mathsf{R}^{n+1} F(X) \longrightarrow \cdots$$

Proof. Since $\mathsf{R}F$ is a triangulated functor, the result follows from applying the cohomology functor H^0. □

Comparing to the classical definition, a great advantage of derived functors in this viewpoint is that they compose in a canonical way.

Proposition 0.8

Consider the additive functors among Abelian categories:

$$\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}''$$

Suppose that the right derived functors $\mathsf{R}F$, $\mathsf{R}F'$ and $\mathsf{R}(F' \circ F)$ all exist. Then there is a natural transformation $\mathsf{R}(F' \circ F) \Rightarrow (\mathsf{R}F') \circ (\mathsf{R}F)$.

Moreover, if $\mathcal{I}$ is an F-injective subcategory of $\mathsf{K}(\mathcal{A})$ and $\mathcal{I}'$ is an F'-injective subcategory of $\mathsf{K}(\mathcal{A}')$ such that $F(\mathrm{Obj}(\mathcal{I})) \subseteq \mathrm{Obj}(\mathcal{I}')$, then $\mathcal{I}$ is $F' \circ F$-injective. And the natural transformation above is an isomorphism:

$$\mathsf{R}(F' \circ F) \cong (\mathsf{R}F') \circ (\mathsf{R}F).$$

Proof. For the first part, the natural transformation $\mathsf{R}(F' \circ F) \Rightarrow (\mathsf{R}F') \circ (\mathsf{R}F)$ is induced by the universal property of left Kan extensions (*check it!*) For the second part, take $I \in \mathrm{Obj}(\mathcal{I})$. Using the construction in Proposition 0.4 we obtain

$$(\mathsf{R}F') \circ (\mathsf{R}F)(QI) = Q'' \circ F' \circ F(I) = \mathsf{R}(F' \circ F)(QI)$$

For $X \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}))$, by choosing quasi-isomorphism $X \to I$ we obtain the isomorphism $(\mathsf{R}F') \circ (\mathsf{R}F)(QX) \cong \mathsf{R}(F' \circ F)(QX)$. Finally check that this is compatible with the natural transformation given above. □

Derived Bi-Functors

The tensor functor $- \otimes -$ and the Hom functor $\mathrm{Hom}(-,-)$ are two typical examples of bi-functors of Abelian categories. Since we are interested in these functors, it is useful to treat the derived bi-functors separately.

Definition 0.9. Let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be triangulated categories. A bi-functor $F\colon \mathcal{K}_1 \times \mathcal{K}_2 \to \mathcal{K}$ is triangulated, if

- F is triangulated in both slots;
- For any $A \in \mathcal{K}_1$ and $B \in \mathcal{K}_2$, the following diagram anti-commutes[2]:

[2]The term is used in [李文威]. It means that the two composite morphisms in the square differ by a sign.

$$\begin{array}{ccc} F(\mathsf{T}_1 A, \mathsf{T}_2 B) & \longrightarrow & \mathsf{T} F(A, \mathsf{T}_2 B) \\ \downarrow & & \downarrow \\ \mathsf{T} F(\mathsf{T}_1 A, B) & \longrightarrow & \mathsf{T}^2 F(A, B) \end{array}$$

The definition of the left/right derived functor of a triangulated bi-functor is essentially identical. We are interested in the cases where the triangulated categories are homotopy categories of Abelian categories.

Now we consider Abelian categories $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$, where $\mathcal{A}$ admits countable products and coproducts. Let $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive bi-functor. Let

$$\mathsf{Ch}_{\oplus} F := \mathrm{Tot}_{\oplus} \circ \mathsf{Ch}^2(F)\colon \mathsf{Ch}(\mathcal{A}_1) \times \mathsf{Ch}(\mathcal{A}_2) \to \mathsf{Ch}(\mathcal{A});$$
$$\mathsf{Ch}_{\Pi} F := \mathrm{Tot}_{\Pi} \circ \mathsf{Ch}^2(F)\colon \mathsf{Ch}(\mathcal{A}_1) \times \mathsf{Ch}(\mathcal{A}_2) \to \mathsf{Ch}(\mathcal{A}).$$

Then induce the triangulated bi-functors $\mathsf{K}_{\oplus} F, \mathsf{K}_{\Pi} F\colon \mathsf{K}(\mathcal{A}_1) \times \mathsf{K}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A})$.

Let $\mathcal{I}_1, \mathcal{I}_2$ be triangulated subcategories of $\mathsf{K}(\mathcal{A}_1), \mathsf{K}(\mathcal{A}_2)$ respectively. We say that $(\mathcal{I}_1, \mathcal{I}_2)$ is F-injective (*resp.* F-projective), if $\mathcal{I}_2$ is $F(A_1, -)$-injective for any $A_1 \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}_1))$, and $\mathcal{I}_1$ is $F(-, A_2)$-injective for any $A_2 \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}_2))$.

Proposition 0.10

Let $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be as above.

1. If $(\mathcal{I}_1, \mathcal{I}_2)$ is F-injective, then $\mathsf{R}F := \mathsf{R}\mathsf{K}_{\Pi} F$ exists. We call it the right derived functor of F;
2. If $(\mathcal{P}_1, \mathcal{P}_2)$ is F-projective, then $\mathsf{L}F := \mathsf{L}\mathsf{K}_{\oplus} F$ exists. We call it the left derived functor of F.

Ext and $\mathsf{R}\,\mathrm{Hom}$

Recall that in *C2.2 Homological Algebra.* we define the $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ to be the n-th right derived functor of $\mathrm{Hom}_{\mathcal{A}}(A, -)$ acting on $B \in \mathrm{Obj}(\mathcal{A})$. If $\mathcal{A}$ has enough injectives or projectives, then $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ is computed by an injective resolution $B \to I^\bullet$ of B or a projective resolution $P^\bullet \to A$ of A. By acyclic assembly lemma, $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ can also be computed as the n-th cohomology of the total complex $\mathrm{Tot}^{\Pi}(\mathrm{Hom}_{\mathcal{A}}(P_\bullet, Q_\bullet))$ using projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$.

Using the derived category, the Ext group can be defined without using injective or projective resolutions:

Definition 0.11. Let $\mathcal{A}$ be an Abelian category. For chain complexes A, B in $\mathsf{Ch}(\mathcal{A})$, we define the **(hyper-)Ext** group as

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) := \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]).$$

This definition gives an obvious multiplication structure on Ext:

$$\begin{array}{ccc} \mathrm{Ext}^n_{\mathcal{A}}(B, C) \times \mathrm{Ext}^m_{\mathcal{A}}(A, B) & \longrightarrow & \mathrm{Ext}^{n+m}_{\mathcal{A}}(A, C) \\ (f, g) & \longmapsto & f[m] \circ g \end{array}$$

In particular it makes $\mathrm{Ext}^\bullet_{\mathcal{A}}(A, A)$ a graded ring for any $A \in \mathrm{Obj}(\mathcal{A})$.

Next we will consider Ext as the right derived functor of Hom bi-functor $\operatorname{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$. It induces the functor on the double complexes:

$$\operatorname{Hom}_{\mathcal{A}}^{\bullet,\bullet}(-,-)\colon \mathsf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Ab}) \times \mathsf{Ch}(\mathsf{Ab}).$$

Define $\mathsf{Ch}\operatorname{Hom}_{\mathcal{A}}(-,-) := \operatorname{Tot}_{\Pi}\operatorname{Hom}_{\mathcal{A}}^{\bullet,\bullet}(-,-)\colon \mathsf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Ab})$. It is not hard to verify that $\mathsf{Ch}\operatorname{Hom}_{\mathcal{A}}$ is naturally isomorphic to the **Hom complex** $\operatorname{Hom}_{\mathcal{A}}^{\bullet}$:

$$\operatorname{Hom}_{\mathcal{A}}^{n}(A,B) := \prod_{k\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(A^{k}, B^{k+n}), \qquad \mathrm{d}_{\mathrm{Hom}}^{n}(f) := \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A.$$

Lemma 0.12

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(A, B[n]) \cong \mathrm{H}^n(\operatorname{Hom}_{\mathcal{A}}^{\bullet}(A,B), \mathrm{d}_{\mathrm{Hom}}^{\bullet}).$$

Proof. Trivial by definition. □

The bi-functor $\mathsf{Ch}\operatorname{Hom}_{\mathcal{A}}$ or $\operatorname{Hom}_{\mathcal{A}}^{\bullet}$ induces the triangulated bi-functor

$$\mathsf{K}\operatorname{Hom}_{\mathcal{A}}\colon \mathsf{K}^{-}(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}^{+}(\mathcal{A}) \to \mathsf{K}^{+}(\mathsf{Ab}).$$

If $\mathcal{A}$ has enough injectives or projectives, then the right derived functor

$$\mathsf{R}\operatorname{Hom}_{\mathcal{A}}\colon \mathsf{D}^{-}(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^{+}(\mathcal{A}) \to \mathsf{D}^{+}(\mathsf{Ab})$$

exists.

Proposition 0.13

Suppose that $\mathcal{A}$ has enough injectives or projectives. For $A \in \operatorname{Obj}(\mathsf{D}^{-}(\mathcal{A}))$ and $B \in \operatorname{Obj}(\mathsf{D}^{+}(\mathcal{A}))$, there exists a canonical isomorphism

$$\mathrm{H}^n\,\mathsf{R}\operatorname{Hom}_{\mathcal{A}}(A,B) \cong \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]).$$

Proof. Taking the right derived functor in the previous lemma and note that the cohomology functor H^n factors through the derived functor. □

Corollary 0.14

Suppose that $\mathcal{A}$ has enough injectives. Let $A, B \in \operatorname{Obj}(\mathcal{A})$ (viewed as complexes concentrated at degree 0). Then there is a canonical isomorphism

$$\operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]) \cong \mathsf{R}^n \operatorname{Hom}(A,-)(B)$$

Therefore the hyper-Ext is a generalisation of the usual Ext.

Tor and $\otimes^{\mathsf{L}}$

In this part we only consider R-modules. For $A, B \in \mathsf{Ch}(R\text{-}\mathsf{Mod})$, from *C3.1 Algebraic Topology* we recall the tensor product of complexes is given by the total complex $A \otimes_R B := \mathrm{Tot}_{\oplus}(A^\bullet \otimes_R B^\bullet)$.

Definition 0.15. For $A, B \in \mathsf{Ch}(R\text{-}\mathsf{Mod})$, the **total tensor product** of A and B is the left derived functor

$$A \otimes_R^{\mathsf{L}} B := \mathsf{L}(- \otimes_R -)(A, B).$$

$\mathsf{L}(- \otimes_R -)\colon \mathsf{D}^-(\mathsf{Mod}\text{-}R) \times \mathsf{D}^-(R\text{-}\mathsf{Mod}) \to \mathsf{D}^-(\mathsf{Ab})$ exists because R-Mod has enough projectives. By taking cohomology we have the **(hyper-)Tor** groups:

$$\mathrm{Tor}_n^R(A, B) := \mathrm{H}_n(A \otimes_R^{\mathsf{L}} B).^3$$

Similar as hyper-Ext, using the theory of derived functors we can verify that the hyper-Tor reduces to the usual Tor on $\mathrm{Obj}(R\text{-}\mathsf{Mod})$ (defined using projective resolutions).

Remark. In general $\mathsf{QCoh}(X)$ does not have enough projectives. We will have to instead use flat resolutions to compute the total tensor product. See later.

Proposition 0.16. Derived Tensor-Hom Adjunction

Let $A \in \mathsf{D}(\mathsf{Mod}\text{-}R)$, $B \in \mathsf{D}(R\text{-}\mathsf{Mod})$, and $C \in \mathsf{D}(\mathsf{Ab})$. There are canonical isomorphisms in $\mathsf{D}(\mathsf{Ab})$:

$$\begin{aligned}\mathsf{R}\,\mathrm{Hom}_{\mathsf{Ab}}(X \otimes_R^{\mathsf{L}} Y, Z) &\cong \mathsf{R}\,\mathrm{Hom}_{\mathsf{Mod}\text{-}R}(X, \mathsf{R}\,\mathrm{Hom}_{\mathsf{Ab}}(Y, Z)) \\ &\cong \mathsf{R}\,\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(Y, \mathsf{R}\,\mathrm{Hom}_{\mathsf{Ab}}(X, Z)).\end{aligned}$$

1 Sheaves of Modules

Let us recall some basic algebraic geometry from *C2.6 Introduction to Schemes*. All rings are commutative with multiplicative identity 1.

Definition 1.1. A **scheme** $(X, \mathcal{O}_X)$ is a locally ringed space such that for any $x \in X$ there exists an open neighbourhood $U \in \mathsf{Top}(X)$ of x such that $(U, \mathcal{O}_X\big|_U) \cong (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.

Example 1.2. A **variety** over a field k is a reduced[4], separated[5], finite type[6] scheme over k. An **affine variety** is a closed subscheme of $\mathbb{A}^n := \operatorname{Spec} k[x_1, ..., x_n]$. A **projective variety** is a reduced closed subscheme of $\mathbb{P}^n := \operatorname{Proj} k[x_0, ..., x_n]$. A **quasi-projective variety** is an open subscheme of a projective variety.

Definition 1.3. Let $(X, \mathcal{O}_X)$ be a scheme. A **sheaf of $\mathcal{O}_X$-modules** F on X is a sheaf $F\colon \mathsf{Top}(X)^{\mathrm{op}} \to \mathsf{Ab}$ such that:

- For any $U \in \mathsf{Top}(X)$, $F(U)$ is a $\mathcal{O}_U$-module;
- The module structure is compatible with restrictions on X.

[3] Cohomology and homology make no difference in algebra. By convention, $\mathrm{H}_n := \mathrm{H}^{-n}$.
[4] i.e. all rings $\mathcal{O}_X(U)$ are reduced rings.
[5] i.e. the diagonal morphism $\Delta\colon X \to X \times_{\operatorname{Spec} k} X$ is a closed immersion.
[6] i.e. quasi-compact and all open affine rings are finite type over k.

The category of $\mathcal{O}_X$-modules is denoted by $\mathcal{O}_X\text{-}\mathsf{Mod}$. It is an Abelian category with enough injectives.

Recall the way we construct the affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ from any ring R. For any R-module M, we can construct the sheaf $\widetilde{M} \in \operatorname{Obj}(\mathcal{O}_{\operatorname{Spec} R}\text{-}\mathsf{Mod})$ in a similar way (see the course notes for details). In particular we have the stalks $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} R$ and the global sections $\widetilde{M}(\operatorname{Spec} R) = M$. For a general scheme X, $\widetilde{M}$ can be constructed from an $\mathcal{O}_X(X)$-module M.

Definition 1.4. Let $F \in \mathcal{O}_X\text{-}\mathsf{Mod}$. We say that F is **quasi-coherent**, if it satisfies any of the following equivalent conditions:

1. F is **locally presented**. That is, for any $x \in X$ there exists a neighbourhood $U \in \mathsf{Top}(X)$ of x such that there exists an exact sequence of the following form:
$$\bigoplus_{i\in I} \mathcal{O}_U \longrightarrow \bigoplus_{j\in J} \mathcal{O}_U \longrightarrow F\big|_U \longrightarrow 0$$
2. For any $x \in X$ there exists an affine neighbourhood $U \cong \operatorname{Spec} R \ni x$ such that $F\big|_U \cong \widetilde{M}$ for some R-module M.
3. There exists an affine open cover $\{U_i\}_{i\in I}$ of X such that $F\big|_{U_i} \cong \widetilde{M_i}$ for R_i-modules M_i, where $\operatorname{Spec} R_i \cong U_i$.

If additionally for each U_i in (3), $F(U_i)$ is a finitely generated $\mathcal{O}_{U_i}$-module, then we say that F is **coherent**. The category of quasi-coherent (*resp.* coherent) sheaves is denoted by $\mathsf{QCoh}(X)$ (*resp.* $\mathsf{Coh}(X)$).

Definition 1.5. Let $F \in \mathcal{O}_X\text{-}\mathsf{Mod}$. We say that F is a **vector bundle** (i.e. locally free of finite rank) if for $x \in X$ there exists an open neighbourhood $U \in \mathsf{Top}(X)$ of x such that $F\big|_U \cong \mathcal{O}_U^{\oplus n}$. The category of vector bundles is denoted by $\mathsf{Vect}(X)$. F is called an **invertible sheaf** (or line bundle) if additionally $n = 1$ for all $x \in X$.

Remark. For a coherent sheaf F on X, if the stalk takes the form $F_x \cong \mathcal{O}_{X,x}^{\oplus n(x)}$ for any $x \in X$, then F is a vector bundle. In particular, $\mathsf{Vect}(X)$ is a full subcategory of $\mathsf{Coh}(X)$ if X is locally Noetherian (i.e. every open affine ring is Noetherian).

Why do we want quasi-coherence?

- $\mathsf{Coh}(X)$ and $\mathsf{QCoh}(X)$ are Abelian categories, but $\mathsf{Vect}(X)$ is not Abelian in general.
- When $X = \operatorname{Spec} R$, $M \mapsto \widetilde{M}$ gives an equivalence of categories $R\text{-}\mathsf{Mod} \simeq \mathsf{QCoh}(X)$.
- Pull-backs preserve quasi-coherence. If X is Noetherian, then push-forwards also preserve quasi-coherence.
- If X is Noetherian, then $\mathsf{QCoh}(X)$ has enough injectives. (*Let's prove it below!*)
- If X and Y are smooth projective varieties, then $\mathsf{Coh}(X) \simeq \mathsf{Coh}(Y)$ implies $X \cong Y$ (*Gabriel–Rosenberg*).

Slogan. Quasi-coherent (*resp.* coherent) sheaves are the analogue of modules (*resp.* finitely generated modules) over a ring.

Functors of Sheaves of Modules

There are some constructions in $\mathcal{O}_X$-Mod.

- **Coproduct**: $\bigoplus_{i\in J} F_i$ is the sheafification of the presheaf $U \mapsto \bigoplus_{i\in J} F_i(U)$;
- **Tensor product**: $F \otimes_{\mathcal{O}_X} G$ is the sheafification of the presheaf $U \mapsto F(U) \otimes_{\mathcal{O}_U} G(U)$.
- **Hom sheaf**: $\mathcal{H}om_{\mathcal{O}_X}(F, G)$ is the presheaf $U \mapsto \operatorname{Hom}_{\mathcal{O}_U}(F\big|_U, G\big|_U)$, which is already a sheaf.
- **Dual sheaf**: $F^\vee := \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

Definition 1.6. Let $f\colon X \to Y$ be a morphism of schemes. Let $F \in \operatorname{Obj}(\mathcal{O}_X\text{-}\mathsf{Mod})$ and $G \in \operatorname{Obj}(\mathcal{O}_Y\text{-}\mathsf{Mod})$.

1. The **direct image** (or push-forward) f_*F of F is a $\mathcal{O}_Y$-module given by $U \mapsto F(f^{-1}(U))$;
2. The **pull-back** f^*G of G is a $\mathcal{O}_X$-module given by $f^*G = f^{-1}(G) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

The key observation is the adjunction $f^* \dashv f_*$: there is a canonical isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*G, F) \cong \operatorname{Hom}_{\mathcal{O}_Y}(G, f_*F).$$

So it is natural to talk about the derived functors of f_* and f^*.

Now let us derive some functors!

Functors	Derived functors	n-th derived functors
Global sections $\Gamma(X,-)\colon \mathsf{Ab}(X) \to \mathsf{Ab}$	$\mathsf{R}\Gamma(X,-)$	Sheaf cohomology $\mathrm{H}^n(X,-)$
$\operatorname{Hom}_{\mathcal{O}_X}(-,-)\colon (\mathcal{O}_X\text{-}\mathsf{Mod})^{\mathrm{op}} \times \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathsf{Ab}$	$\mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(-,-)$	Ext group $\operatorname{Ext}^n_X(-,-)$
$\mathcal{H}om_{\mathcal{O}_X}(-,-)\colon (\mathcal{O}_X\text{-}\mathsf{Mod})^{\mathrm{op}} \times \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathcal{O}_X\text{-}\mathsf{Mod}$	$\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(-,-)$	Ext sheaf $\mathcal{E}xt^n_X(-,-)$
$-\otimes_{\mathcal{O}_X} -\colon \mathcal{O}_X\text{-}\mathsf{Mod} \times \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathcal{O}_X\text{-}\mathsf{Mod}$	$-\otimes^{\mathsf{L}}_{\mathcal{O}_X} -$	Tor group $\operatorname{Tor}^X_n(-,-)$
$f_*\colon \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathcal{O}_Y\text{-}\mathsf{Mod}$	$\mathsf{R}f_*$	Higher direct image $\mathsf{R}^n f_*$
$f^*\colon \mathcal{O}_Y\text{-}\mathsf{Mod} \to \mathcal{O}_X\text{-}\mathsf{Mod}$	$\mathsf{L}f^*$	$\mathsf{L}_n f^*$

Derived Categories of Coherent Sheaves

We will always assume that X is Noetherian[7]. A good new and a bad news.

Proposition 1.7

Let X be a Noetherian scheme. Then $\mathsf{QCoh}(X)$ has enough injectives.

Proof. [HartsAG, Cor III.3.6] Cover X with a finite number of affine opens $U_i = \operatorname{Spec} A_i$, and let $F|_{U_i} = \widetilde{M_i}$ for each i. Embed M_i in an injective A_i-module I_i. For each i, let $f : U_i \to X$ be the inclusion, and let $G = \bigoplus_i f_*(\widetilde{I_i})$. For each i we have an injective map of sheaves $F|_{U_i} \to \widetilde{I_i}$. Hence we obtain a map $F \to f_*(\widetilde{I_i})$. Taking the direct sum over i gives a map $F \to G$ which is clearly injective. Check that G is flasque[8] and quasi-coherent. G is an injective object in $\mathsf{QCoh}(X)$. □

[7] i.e. quasi-compact and every open affine ring is Noetherian.
[8] i.e. restriction maps of F are surjective.

Remark. Alternatively it can also be shown that $\mathsf{QCoh}(X)$ is a **Grothendieck category** (see [李文威, §2.10]), thus having enough injectives.

In general $\mathsf{Coh}(X)$ does not have enough injectives. Think of $X = \operatorname{Spec}\mathbb{Z}$, where $\mathsf{Coh}(X)$ is the category of finitely generated Abelian groups. Instead of $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$, we instead work with the full subcategory $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ of $\mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)$:

$$\mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)) := \left\{F \in \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)\colon \ \mathrm{H}^n(F) \in \mathrm{Obj}(\mathsf{Coh}(X)); \ \mathrm{H}^i(F) = 0 \text{ for } |i| \gg 0\right\}.$$

In general for a full Abelian subcategory $\mathcal{A} \subseteq \mathcal{B}$, the derived categories $\mathsf{D}(\mathcal{A})$ and $\mathsf{D}_{\mathcal{A}}(\mathcal{B})$ could be quite different. However we have the following

Proposition 1.8

Let X be a Noetherian scheme. The natural functor $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \to \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)$ defines a triangulated equivalence of categories

$$\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X).$$

Proof. [Huyb, Prop 3.5] It is clear that $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \to \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)$ is fully faithful. It suffices to show essential surjectivity. Consider a bounded complex of quasi-coherent sheaves with coherent cohomology:

$$0 \longrightarrow F^n \longrightarrow \cdots \longrightarrow F^m \longrightarrow 0$$

By induction suppose F^j is coherent for $j > i$. Consider the surjections $\mathrm{d}^i\colon F^i \to \operatorname{im}\mathrm{d}^i \subseteq F^{i+1}$ and $\ker \mathrm{d}^i \to \mathrm{H}^i(F^\bullet)$. We can find coherent subsheaves of $F^i_1 \subseteq F^i$ and $F^i_2 \subseteq \ker\mathrm{d}^i \subseteq F^i$ such that the restrictions of the above morphisms are still surjective ([HartsAG, Ex II.5.15]). Now replace F^i by its subsheaf generated by F^i_1 and F^i_2, and let F^{i-1} be the preimage under d^{i-1} of the new F^i. Clearly the inclusions induce a quasi-isomorphism of the new complex with the old one and now F^i is also coherent. □

So we can resolve a coherent sheaf by quasi-coherent sheaves injective in $\mathsf{QCoh}(X)$ in order to compute $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$.

Derived Functors of Coherent Sheaves

In this part we address some technical issues in passing the functors from $\mathcal{O}_X\text{-}\mathsf{Mod}$ to $\mathsf{Coh}(X)$. We follow [Huyb §3.3]. A lot of relevant results are scattered in Chapter III of [HartsAG]...

Theorem 1.9. Grothendieck Vanishing Theorem

Let X be a Noetherian topological space of dimension n. Then $\mathrm{H}^i(X, F) = 0$ for all $F \in \mathrm{Obj}(\mathsf{Ab}(X))$ and $i > n$.

Proof. See [HartsAG Thm III.2.7]. □

Theorem 1.10

Let F be a coherent sheaf on a scheme X which is proper (e.g. projective) over a field k. Then $\mathrm{H}^i(X, F)$ is finite dimensional over k for all i.

Proof. See [HartsAG Thm III.5.2]. □

Corollary 1.11

Let X be a projective variety over a field k. The global section functor $\Gamma(X, -)$ is a left exact functor $\mathsf{Coh}(X) \to k\text{-}\mathsf{Mod}^{\mathrm{fd}}$. The right derived functor $\mathsf{R}\Gamma$ can be computed via the composition $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \hookrightarrow \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X) \to \mathsf{D}^{\mathrm{b}}(k\text{-}\mathsf{Mod})$.

Theorem 1.12

1. Let $f\colon X \to Y$ be a morphism of Noetherian schemes. Let F be a quasi-coherent sheaf over X. The higher direct images $\mathsf{R}^i f_*(F) = 0$ for $i > \dim X$.
2. Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. Let F be a coherent sheaf over X. The higher direct images $\mathsf{R}^i f_*(F)$ are also coherent for all i.

Proof. See [HartsAG Thm III.8.1 III.8.8]. □

Corollary 1.13

Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. The direct image $f_*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(Y)$ induces the right derived functor $\mathsf{R}f_*\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$.

Remark. For the derived functors $- \otimes^{\mathsf{L}} -$ and $\mathsf{R}\mathcal{H}om$ in D^{b}, we must be able to take bounded resolutions. This is possible when X is smooth projective. We discuss them in the next section.

Lemma 1.14. Projection Formula

Let $f\colon X \to Y$ be a proper morphism of projective schemes. For $F \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ and $E \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$, there is a canonical isomorphism

$$\mathsf{R}f_*(F) \otimes^{\mathsf{L}} E \cong \mathsf{R}f_*(F \otimes^{\mathsf{L}} \mathsf{L}f^* E).$$

This is a consequence of the classical projective formula $f_*F \otimes E \cong f_*(F \otimes f^*E)$ where E is a vector bundle and F is an arbitrary $\mathcal{O}_X$-module.

2 Coherent Sheaves on a Smooth Projective Variety

Smoothness

Let k be an algebraically closed field. Recall that in *C3.4 Algebraic Geometry* we define the non-singular points of a quasi-projective variety by counting the dimension of (co)tangent space at that point:

Definition 2.1. A scheme X is **non-singular** (or regular)[9] at $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring. That is, $\dim_{\mathcal{O}_{X,x}/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$. X is non-singular if it is non-singular at all points[10].

The non-singularity can be characterised by Kähler differentials, which is the algebraic analogue of the cotangent bundle.

Proposition 2.2

Let X be an irreducible variety over k. Then X is regular if and only if the sheaf of Kähler differentials $\Omega_{X/k}$ is a vector bundle over X of dimension $n = \dim X$.

Proof. See [HartsAG Thm II.8.15]. □

Definition 2.3. Let X be a non-singular irreducible variety over k. Let $n = \dim X$. We define the

- **tangent sheaf/bundle** $\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$, which is a vector bundle of rank n;
- **canonical sheaf/bundle** $\omega_X := \bigwedge^n \Omega_{X/k}$, which is a line bundle.

Perfect Complexes

Definition 2.4. Let $F \in \mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X))$. We say that F is a **strictly perfect complex**, if F is quasi-isomorphic to a bounded complex of vector bundles on X. We say that F is a **perfect complex** if there exists an affine cover $\{U_i\}_{i \in I}$ of X such that each $F\big|_{U_i}$ is quasi-isomorphic to some strictly perfect complex F_i on U_i.

The perfect complexes form a full subcategory $\mathsf{Perf}(X)$ of $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$.

Proposition 2.5. Smoothness via Perfect Complexes

Suppose that X is a Noetherian scheme. Then X is regular if and only if the inclusion $\mathsf{Perf}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ is an equivalence of categories.

Proof. Idea: On a regular scheme X, any coherent sheaf F admits a locally free resolution of length $\dim X$. This is the generalisation of the affine result: $\operatorname{Spec} R$ is an n-dimensional regular affine variety if and only if every (finitely generated) R-module M admits a (finitely generated) projective resolution of length n. □

Remark. For a general variety X, we may introduce the quotient category (*localisation?*)

$$\mathsf{Sing}(X) := \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) / \mathsf{Perf}(X)$$

which measures how singular X is. Of course $\mathsf{Sing}(X)$ is trivial if X is regular.

By passing to $\mathsf{Perf}(X)$ we will be able to define the bounded version of $\mathsf{R}\mathcal{H}om$ and $\otimes^{\mathsf{L}}$ for coherent sheaves when X is a smooth projective variety. In particular, for $F \in \mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X))$, the **derived dual**

$$F^{\vee} := \mathsf{R}\mathcal{H}om(F, \mathcal{O}_X) \in \mathsf{D}^{+}\mathsf{QCoh}(X)$$

[9]It is bad to use the term *smooth* here, as it is reserved for a property of morphisms.

[10]Equivalently at all closed points, because the stalk at any non-closed point is a localisation of the stalk at a closed point, and localisation preserves regular local rings.

is in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ when X is regular.

Serre Duality

Theorem 2.6. Serre Duality

Let X be a n-dimensional smooth projective variety over k with canonical sheaf ω_X. For $F \in \operatorname{Obj}(\mathsf{Vect}(X))$, there are functorial isomorphisms of vector spaces

$$\mathrm{H}^i(X, F)^\vee \cong \operatorname{Ext}^{n-i}_X(F, \omega_X) \cong \mathrm{H}^{n-i}(X, F^\vee \otimes_{\mathcal{O}_X} \omega_X).$$

Proof. See [HartsAG §III.7]. The second isomorphism follows from the general facts $\operatorname{Ext}^n_X(E \otimes_{\mathcal{O}_X} F, G) \cong \operatorname{Ext}^n(E, F^\vee \otimes_{\mathcal{O}_X} G)$ (here F needs to be a vector bundle) and $\operatorname{Ext}^n(\mathcal{O}_X, F) \cong \mathrm{H}^n(X, F)$ for $\mathcal{O}_X$-modules E, F, G. □

Remark. If we take $F = \Omega^p := \bigwedge^p \Omega_{X/k}$ and note that $\Omega^{n-p} \cong (\Omega^p)^\vee \otimes_{\mathcal{O}_X} \omega_X$ ([HartsAG Ex II.5.16.(b)]), then Serre duality takes the form

$$\mathrm{H}^q(X, \Omega^p)^\vee \cong \mathrm{H}^{n-q}(X, \Omega^{n-p}),$$

which is known in complex geometry.

Corollary 2.7

Let X be a n-dimensional smooth projective variety over k. Then $\mathsf{Coh}(X)$ has **global homological dimension** n. That is, $\operatorname{Ext}^i_X(F, G) = 0$ for $i > n$ and any coherent sheaves F, G.

Remark. In particular, for a smooth projective curve C, $\mathsf{Coh}(C)$ has global homological dimension 1. It can be proven that every $F \in \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(C)$ is quasi-isomorphic to its cohomology:

$$F \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(F)[-i].$$

Serre Functor

Let us rephrase Serre duality using some category theory.

Definition 2.8. Let $\mathcal{A}$ be a k-linear category. A **Serre functor** $S\colon \mathcal{A} \to \mathcal{A}$ is a k-linear equivalence such that for $A, B \in \operatorname{Obj}(\mathcal{A})$ there exists a functorial isomorphism of vector spaces

$$\operatorname{Hom}_{\mathcal{A}}(A, B) \cong \operatorname{Hom}_{\mathcal{A}}(B, S(A)).$$

Lemma 2.9

Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories with finite-dimensional Hom spaces. Suppose that they admit Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ respectively. Then any k-linear equivalence $F\colon \mathcal{A} \to \mathcal{B}$ commutes with the Serre functors: $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$.

Proof. This is an application of the Yoneda lemma: since F is fully faithful, one has for any two objects $A, B \in \mathcal{A}$

$$\operatorname{Hom}(A, S_{\mathcal{A}}B) \cong \operatorname{Hom}(FA, FS_{\mathcal{A}}B), \qquad \operatorname{Hom}(B, A) \cong \operatorname{Hom}(FB, FA).$$

Together with the two isomorphisms

$$\operatorname{Hom}(A, S_{\mathcal{A}}B) \cong \operatorname{Hom}(B, A)^{\vee}, \qquad \operatorname{Hom}(FB, FA) \cong \operatorname{Hom}(FA, S_{\mathcal{B}}FB)^{\vee},$$

this yields a functorial isomorphism

$$\operatorname{Hom}(FA, FS_{\mathcal{A}}B) \cong \operatorname{Hom}(FA, S_{\mathcal{B}}FB).$$

Using the hypothesis that F is an equivalence and, in particular, that any object in $\mathcal{B}$ is isomorphic to some $F(A)$, one concludes that there exists a functor isomorphism $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$. □

Remark. If $\mathcal{A}, \mathcal{B}$ are triangulated categories, then the Serre functors are exact and triangulated.

In particular, Serre functors are useful in inverting adjunction pairs:

Corollary 2.10

Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a k-linear functor. Then

$$G \dashv F \implies F \dashv S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}.$$

Proof. For $A \in \operatorname{Obj}(\mathcal{A})$ and $B \in \operatorname{Obj}(\mathcal{B})$,

$$\operatorname{Hom}_{\mathcal{A}}(A, S_{\mathcal{A}}GS_{\mathcal{B}}^{-1}B) \cong \operatorname{Hom}_{\mathcal{A}}(GS_{\mathcal{B}}^{-1}B, A)^{\vee} \cong \operatorname{Hom}_{\mathcal{B}}(S_{\mathcal{B}}^{-1}B, FA)^{\vee} \cong \operatorname{Hom}_{\mathcal{B}}(FA, B)$$ □

Serre functors gain their name from Serre duality. Indeed, let X be a smooth projective variety. We define the the functor

$$S_X\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X), \qquad F \longmapsto F \otimes_{\mathcal{O}_X} \omega_X[\dim X].$$

Proposition 2.11

The functor S_X defined above is a Serre functor.

Proof. Let $n = \dim X$. let E, F be vector bundles over X. By Serre duality we have

$$\operatorname{Ext}^i_X(E, F) \cong \mathrm{H}^i(X, E^{\vee} \otimes F) \cong \mathrm{H}^{n-i}(X, E \otimes F^{\vee} \otimes \omega_X)^{\vee} \cong \operatorname{Ext}^{n-i}_X(F, E \otimes \omega_X)^{\vee}.$$

Using Corollary 0.14 we obtain

$$\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(E, F[i]) \cong \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(F[i], E \otimes \omega_X[n])^{\vee} \cong \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(F[i], S_X(E))^{\vee}.$$

Therefore for any $E, F \in \operatorname{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X))$, we have

$$\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(E, F) \cong \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(F, S_X(E))^{\vee}.$$ □

Grothendieck–Verdier Duality

The target is to generalise Serre duality to a relative version. Let $f\colon X \to Y$ be a morphism of smooth projective varieties. We define the **relative dimension** $\dim f := \dim X - \dim Y$ and the **relative dualising bundle** $\omega_f := \omega_X \otimes_{\mathcal{O}_X} f^*\omega_Y^{-1}$.

It is impossible to find a right adjoint to the direct image functor $f_*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(Y)$, because we have the adjunction $f^* \dashv f_*$ on the Abelian categories $\mathsf{Coh}(X)$ and $\mathsf{Coh}(Y)$. However it is possible after passing to the derived categories. We can construct $\mathsf{L}f^* \dashv \mathsf{R}f_* \dashv f^!$ by Serre functors.

Theorem 2.12. Grothendieck–Verdier Duality

Let $f\colon X \to Y$ be a morphism of smooth projective varieties. Then the right adjoint of $\mathsf{R}f_*\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ exists and is given by

$$f^!(F) := \mathsf{L}f^*(F) \otimes_{\mathcal{O}_X} \omega_f[\dim f].$$

Proof. By the previous part it suffices to take $f^! := S_X \circ \mathsf{L}f^* \circ S_Y^{-1}$. □

Grothendieck–Verdier duality has a more general form, which is a functorial isomorphism

$$\mathsf{R}f_* \circ \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(F, \mathsf{L}f^*(E) \otimes_{\mathcal{O}_X} \omega_f[\dim f]) \cong \mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*(F), E)$$

for $F \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ and $E \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$.

3 Reconstruction from Derived Categories

Ampleness

Let us first recall the structure of invertible sheaves on the projective space $\mathbb{P}^n$. Let L be an invertible sheaf on a scheme X. It is called invertible because the tensor operation with the dual sheaf gives

$$L \otimes_{\mathcal{O}_X} L^\vee = L \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(L, L) \cong \mathcal{O}_X.$$

Therefore the set of invertible sheaves forms a group $\operatorname{Pic} X$ under the tensor operation, called the **Picard group** of X. For $X = \mathbb{P}^n_k = \operatorname{Proj} S$, where $S = k[x_0, ..., x_n]$, we have the **twisting sheaf** on $\mathbb{P}^n_k$:

$$\mathcal{O}(1) := \widetilde{S[1]}, \qquad S[1] \text{ is a graded } S\text{-module with } S[1]_d = S_{d+1}.$$

Let $\mathcal{O}(0) := \mathcal{O}_{\mathbb{P}^n_k}$, $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$ for $n > 0$ and $\mathcal{O}(n) := \mathcal{O}(-n)^\vee$ for $n < 0$. It can be proven that $\mathcal{O}(n) = \widetilde{S[n]}$. Then we have a subgroup of $\operatorname{Pic} \mathbb{P}^n_k$ isomorphic to $\mathbb{Z}$. In fact it can be proven (e.g. using divisors) that all invertible sheaves on $\mathbb{P}^n_k$ are in this form. So $\operatorname{Pic} \mathbb{P}^n_k \cong \mathbb{Z}$.

By definition, the gloval sections of $\mathcal{O}(n)$ are generated by the homogeneous elements in S of degree n. In particular, the twisting sheaf $\mathcal{O}(1)$ has global sections generated by $x_0, ..., x_n$, and $\mathcal{O}(n)$ has no global sections for $n < 0$.

Remark. For general X, using Čech cohomology it can be proven that $\operatorname{Pic} X \cong \check{\mathrm{H}}^1(X, \mathcal{O}_X^\times)$, where $\mathcal{O}_X^\times$ is the **sheaf of invertible functions**, that is, $\mathcal{O}_X^\times(U)$ is the multiplicative group of $\mathcal{O}_X(U)$ for each $U \in \mathsf{Top}(X)$.

Lemma 3.1. Euler Exact Sequence

There is a short exact sequence of sheaves on $X = \mathbb{P}^n_k$:

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \mathcal{O}_X(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Proof. See [HartsAG Thm II.8.13]. □

Definition 3.2. Let X be a scheme over the field k, and L be an invertible sheaf on X. We say that L is **very ample** (relative to $\operatorname{Spec} k$), if there exists a (locally closed) immersion $\iota \colon X \to \mathbb{P}^n_k$ such that $\iota^*(\mathcal{O}(1)) \cong L$. This is equivalent to saying that L is generated by the global sections $s_0, \ldots, s_n$, where $s_i := \iota^*(x_i)$.

Lemma 3.3

Let X be a projective scheme over k and let L be a very ample invertible sheaf on X. Let $F \in \operatorname{Obj}(\mathsf{Coh}(X))$. Then for $n \gg 0$, $F \otimes_{\mathcal{O}_X} L^{\otimes n}$ is generated by finitely many global sections.

Proof. See [HartsAG Thm II.5.17]. □

Definition 3.4. Let X be a Noetherian scheme, and L be an invertible sheaf on X. We say that L is **ample** if for any $F \in \operatorname{Obj}(\mathsf{Coh}(X))$, there exists $n_0 > 0$ such that for $n \geqslant n_0$, $F \otimes_{\mathcal{O}_X} L^{\otimes n}$ is generated by global sections.

Theorem 3.5

Let X be a projective variety over k, and L be an invertible sheaf on X. The following are equivalent:

- L is ample;
- $L^{\otimes m}$ is ample for some $m > 0$;
- $L^{\otimes m}$ is very ample (relative to $\operatorname{Spec} k$) for some $m > 0$.

Proof. See [HartsAG II.7.5, II.7.6]. □

Definition 3.6. Let X be a non-singular variety with canonical bundle ω_X and anti-canonical bundle $\omega_X^\vee$. X is called a

- **Fano variety**, if $\omega_X^\vee$ is ample;
- **Calabi–Yau variety**, if $\omega_X = \mathcal{O}_X$;
- **anti-Fano variety**[11], if ω_X is ample.

Remark. Consider compact Kähler manifolds which admit projective embeddings. By the celebrated Calabi–Yau theorem, the three cases above correspond to Kähler metrics with positive, flat, and negative Ricci curvature respectively.

[11] This non-standard terminology is used in [Bock].

Remark. The projective space $\mathbb{P}^n$ is Fano because $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$ ([HartsAG II.8.13, II.8.20.1]), and $\mathcal{O}(n)$ is ample if and only if $n > 0$.

Remark. For a smooth projective curve C with genus g, C is Fano if $g = 0$, Calabi–Yau if $g = 1$ (i.e. elliptic curve), and anti-Fano if $g > 1$.

Lemma 3.7

Let X be a projective variety over k, and L be an ample invertible sheaf on X. Then $X \cong \operatorname{Proj}\Gamma_*(X, L^{\otimes m})$ for some $m \in \mathbb{Z}_+$, where $\Gamma_*(X, L)$ is the graded ring $\bigoplus_{d=0}^{\infty} \Gamma(X, L^{\otimes d})$.

Proof. See `math.stackexchange.com/questions/57775` or (Stacks Project Lemma 28.26.9). □

Bondal–Orlov Reconstruction Theorem

The target is to explain the idea of the following result. We follows [Huyb §4.1].

Theorem 3.8. Bondal–Orlov Reconstruction Theorem

Suppose that X and Y are smooth projective varieties over k. If X is Fano or anti-Fano, and $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$, then $X \cong Y$.

The proof can be divided into the following steps:

1. Identify point-like and invertible objects in the derived categories which generalise the invertible sheaves and skyscraper sheaves on the variety.
2. Since point-like objects and invertible objects are preserved under the equivalence $F\colon \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \to \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$, prove that $\mathcal{O}_X$ is mapped to $\mathcal{O}_Y$, and that Y is also Fano or anti-Fano.
3. Prove the graded ring isomorphism $\bigoplus_{d=0}^{\infty} \Gamma(X, \omega_X^{\otimes d}) \cong \bigoplus_{d=0}^{\infty} \Gamma(Y, \omega_Y^{\otimes d})$.
4. By ampleness of ω_X (or $\omega_X^{\vee}$), X can be reconstructed as $\operatorname{Proj}\left(\bigoplus_{d=0}^{\infty} \Gamma(X, \omega_X^{\otimes d})\right)$. Thus conclude that $X \cong Y$.

Definition 3.9. Let $\mathcal{K}$ be a k-linear triangulated category with a Serre functor S. An object $P \in \operatorname{Obj}(\mathcal{K})$ is called **point-like** of codimension d if

1. $S(P) \cong P[d]$;
2. $\operatorname{Hom}_{\mathcal{K}}(P, P[i]) = 0$ for $i < 0$;
3. $\kappa(P) := \operatorname{Hom}_{\mathcal{K}}(P, P)$ is a field.

Remark. Consider $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ for smooth projective variety k with the Serre functor S_X. For $x \in X$, the skyscraper sheaf $\underline{\kappa(x)}$ of the residue field $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ supported at x is a point-like object of codimension $\dim X$ in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$. This explains the name. Moreover, we shall show that every point-like object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ arises from them for X Fano or anti-Fano.

Lemma 3.10

Suppose that X is a smooth projective varieties over k. If X is Fano or anti-Fano, then every point-like object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ is isomorphic to $\underline{\kappa(x)}[m]$, where $x \in X$ is a closed point and $m \in \mathbb{Z}$.

Proof. See [Huyb 4.5, 4.6]. □

Remark. This is certain not true when X is not Fano or anti-Fano. For example, if X is Calabi–Yau, then $\mathcal{O}_X$ is a point-like object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$.

Definition 3.11. Let $\mathcal{K}$ be a k-linear triangulated category with a Serre functor S. An object $L \in \mathrm{Obj}(\mathcal{K})$ is called **invertible** if for any point-like object $P \in \mathrm{Obj}(\mathcal{K})$ there exists $n \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{\mathcal{K}}(L, P[i]) = \begin{cases} \kappa(P), & i = n; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.12

Suppose that X is a smooth projective varieties over k. Every invertible object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ is of the form $L[m]$ where L is an invertible sheaf on X and $m \in \mathbb{Z}$.

Conversely, if X is Fano or anti-Fano, then $L[m]$ in an invertible object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ for L invertible sheaf on X and $m \in \mathbb{Z}$.

Proof. See [Huyb Prop 4.9]. □

Lemma 3.13

Suppose that X and Y are smooth projective varieties over k. If $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$, then $\dim X = \dim Y$.

Proof. For a closed point $x \in X$, the skyscraper sheaf

$$\underline{\kappa(x)} \cong \underline{\kappa(x)} \otimes \omega_X = S_X(\underline{\kappa(x)})[-\dim X].$$

Under the equivalence $F \colon \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \to \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$,

$$F(\underline{\kappa(x)}) \cong F(S_X(\underline{\kappa(x)})[-\dim X]) \cong S_Y(F(\underline{\kappa(X)}))[-\dim X] \cong F(\underline{\kappa(x)}) \otimes \omega_Y[\dim Y - \dim X].$$

Taking the cohomology sheaf of the bounded complex $F(\underline{\kappa(x)})$ and using that ω_Y commutes with cohomology, we have

$$\mathcal{H}^i(F(\underline{\kappa(x)})) \cong \mathcal{H}^{i+\dim Y - \dim X}(F(\underline{\kappa(x)})) \otimes \omega_Y.$$

By looking at the maximal and minimal i such that $\mathcal{H}^i(F(\underline{\kappa(x)})) \neq 0$, we deduce that $\dim X = \dim Y$. □

Proof of Bondal–Orlov theorem assuming above lemmata.

Let $F \colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ be an exact equivalence. It is clear that F preserves invertible objects. Then $F(\mathcal{O}_X)$ is invertible and is of the form $L[m]$ for some invertible sheaf L on Y. Then $F' := T^{-m} \circ (L^\vee \otimes -) \circ F$ is another exact equivalence $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ such that $F'(\mathcal{O}_X) \cong \mathcal{O}_Y$. We simply replace F by F'.

Assume that ω_X is ample (the other case is similar). Let $n = \dim X = \dim Y$. We have for $d \in \mathbb{N}$,

$$F(\omega_X^{\otimes d}) = F(S_X^d(\mathcal{O}_X))[-dn] \cong S_Y^k(F(\mathcal{O}_X))[-dn] \cong S_Y^d(\mathcal{O}_Y)[-dn] = \omega_Y^d$$

and hence $\Gamma(X, \omega_X^d) = \mathrm{Hom}(\mathcal{O}_X, \omega_X^d) \cong \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^d) = \Gamma(Y, \omega_Y^d)$. This induces an graded ring isomorphism $\bigoplus_{d=0}^\infty \Gamma(X, \omega_X^d) \cong \bigoplus_{d=0}^\infty \Gamma(Y, \omega_Y^d)$, where the multiplication is given by

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_X, \omega_X^{d_1}) \times \mathrm{Hom}(\mathcal{O}_X, \omega_X^{d_2}) &\longrightarrow \mathrm{Hom}(\mathcal{O}_X, \omega_X^{d_1+d_2}) \\ (s_1, s_2) &\longmapsto S_X^{d_1}(s_2)[-d_1 n] \circ s_1 \end{aligned}$$

Note that ω_X is ample implies that $\omega_X^{\otimes m}$ is very ample for some $m > 0$, which implies that $X \cong \mathrm{Proj}\left(\bigoplus_{d=0}^\infty \Gamma(X, \omega_X^{\otimes md})\right)$ If $\omega_Y^{\otimes m}$ is also very ample, then we may conclude that

$$X \cong \mathrm{Proj}\left(\bigoplus_{d=0}^\infty \Gamma(X, \omega_X^{\otimes md})\right) \cong \mathrm{Proj}\left(\bigoplus_{d=0}^\infty \Gamma(Y, \omega_Y^{\otimes md})\right) \cong Y.$$

Finally we prove that $\omega_Y^{\otimes m}$ is very ample. The idea is that this is equivalent to that the Zariski topology on Y has a basis of the form $\left\{V_\beta \colon \beta \in \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^{\otimes md}),\ d \in \mathbb{Z}\right\}$, where $V_\beta := \left\{y \in Y \colon \alpha_y^* \neq 0\right\}$, and $\alpha_y^* \colon \mathrm{Hom}(\omega_Y^{\otimes md}, \underline{\kappa(y)}) \to \mathrm{Hom}(\mathcal{O}_Y, \underline{\kappa(y)})$ is the induced map $f \mapsto f \circ \alpha$. But the equivalence F induces a homeomorphism $X \to Y$, which maps U_α in X to $V_{F(\alpha)}$ in Y. This implies that $\omega_Y^{\otimes m}$ is very ample. □

Remark. By Bondal–Orlov theorem, a smooth projective curve with genus $g \neq 1$ is completely determined by its derived category of coherent sheaves. For elliptic curves, this is also true.

Theorem 3.14

Suppose that X and Y are smooth projective curves over k. If $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$, then $X \cong Y$.

Proof. See [Huyb Cor 5.46]. □

The theorem tells something more about the autoequivalence group of $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$.

Corollary 3.15

uppose that X is a smooth projective variety which is Fano or anti-Fano. Then

$$\mathrm{Aut}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)) \cong \mathbb{Z} \times (\mathrm{Aut}\, X \ltimes \mathrm{Pic}\, X).$$

Proof. See [Huyb Prop 4.17]. □

Fourier–Mukai Transforms

In analysis, an integral transform Φ_K from $\mathbb{R}^n$ to $\mathbb{R}^n$ with kernel $K\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{C}$ takes the form

$$\Phi_K(f)(p) := \int_{\mathbb{R}^n} f(x)K(x,p)\,\mathrm{d}x.$$

For example Φ_K is the Fourier transform when $K(x,p) = \dfrac{1}{2\pi}\,\mathrm{e}^{-\mathrm{i}x\cdot p}$. We generalise this idea to algebraic geometry to produce a class of functors between the derived categories.

Definition 3.16. Let X and Y be smooth projective varieties over k. Let $\pi_X\colon X\times_k Y \to X$ and $\pi_Y\colon X\times_k Y \to Y$ be the projection maps. For $E \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times_k Y)$, we define the **integral transform** $\Phi^E_{X\to Y}$ with kernel E to be the functor

$$\Phi^E_{X\to Y}\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y), \qquad F \longmapsto \mathsf{R}(\pi_Y)_*(\pi_X^*(F)\otimes^{\mathsf{L}} E).$$

If $\Phi^E_{X\to Y}$ is an exact equivalence of categories, then it is called a **Fourier–Mukai transform**.

A lot of derived functors we have known can be expressed as an integral transform:

- The identity functor $\mathrm{id}\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ is isomorphic to $\Phi^{\mathcal{O}_\Delta}_{X\to X}$, where $\mathcal{O}_\Delta := \Delta_*\mathcal{O}_X$ is the push-forward by the diagonal morphism $\Delta\colon X \to X\times X$.
- For $E\in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$, the derived tensor product $-\otimes^{\mathsf{L}}-$ is isomorphic to $\Phi^{\Delta_*E}_{X\to X}$.
- Let $f\colon X\to Y$ be a morphism. $\Gamma_f \subseteq X\times Y$ is the graph of f. Then $\mathcal{O}_{\Gamma_f} \in \mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times Y))$. The derived direct image $\mathsf{R}f_*$ is isomorphic to $\Phi^{\mathcal{O}_{\Gamma_f}}_{X\to Y}$ and the derived pull-back $\mathsf{L}f^*$ is isomorphic to $\Phi^{\mathcal{O}_{\Gamma_f}}_{Y\to X}$.

Proposition 3.17

Let $\Phi^E_{X\to Y}\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)\to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ be an integral transform with kernel $E\in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times Y)$. Then it admits left and right adjoints, respectively given by $\Phi^{E^\vee\otimes\pi_Y^*\omega_Y[\dim Y]}_{Y\to X}$ and $\Phi^{E^\vee\otimes\pi_X^*\omega_X[\dim X]}_{Y\to X}$, where $E^\vee := \mathsf{R}\mathcal{H}om(E,\mathcal{O}_{X\times Y})$.

Proof. This is a nice application of the Grothendieck–Verdier duality. For $G\in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ and $F\in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$,

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(\Phi^{E^\vee\otimes\pi_Y^*\omega_Y[\dim Y]}_{Y\to X}(F), G)\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(\mathsf{R}(\pi_X)_*(\pi_Y^*F\otimes^{\mathsf{L}} E^\vee\otimes\pi_Y^*\omega_Y[\dim Y]), G)\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times Y)}(\pi_Y^*F\otimes^{\mathsf{L}} E^\vee\otimes\pi_Y^*\omega_Y[\dim Y], \pi_X^!G)\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times Y)}(\pi_Y^*F\otimes^{\mathsf{L}} E^\vee\otimes\pi_Y^*\omega_Y[\dim Y], \mathsf{L}\pi_X^*G\otimes\pi_Y^*\omega_Y[\dim Y])\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times Y)}(\pi_Y^*F\otimes^{\mathsf{L}} E^\vee, \mathsf{L}\pi_X^*G)\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X\times Y)}(\mathsf{L}\pi_Y^*F, E\otimes^{\mathsf{L}}\pi_X^*G)\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)}(F, \mathsf{R}(\pi_Y)_*(E\otimes^{\mathsf{L}}\pi_X^*G))\\
&\quad= \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)}(F, \Phi^E_{X\to Y}(G)).
\end{aligned}$$

Therefore we have $\Phi^{E^\vee\otimes\pi_Y^*\omega_Y[\dim Y]}_{Y\to X} \dashv \Phi^E_{X\to Y}$. For the right adjoint of $\Phi^E_{X\to Y}$, we can use

Corollary 2.10. ☐

Proposition 3.18

For $E \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X \times Y)$ and $F \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y \times Z)$, define

$$F \circ E := \mathsf{R}(\pi_{XZ})_*(\pi_{XY}^* E \otimes^{\mathsf{L}} \pi_{YZ}^* F),$$

where $\pi_{XY}.\pi_{YZ}.\pi_{XZ}$ are projections from $X \times Y \times Z$ to $X \times Y$, $Y \times Z$, and $X \times Z$ respectively. Then there is a natural isomorphism of functors

$$\Phi^{F \circ E}_{X \to Z} \cong \Phi^{F}_{Y \to Z} \circ \Phi^{E}_{X \to Y}.$$

Proof. The checking is straightforward. See [Huyb Prop 5.10]. ☐

There is a famous difficult result due to Orlov:

Theorem 3.19. Orlov's Theorem

Let X and Y be smooth projective varieties and let $F\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ be a fully faithful exact functor. There exists a unique $E \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X \times Y)$ such that $F \cong \Phi^{E}_{X \to Y}$.

In particular, if F is an equivalence, then it is isomorphic to a Fourier–Mukai transform with a unique kernel.

Corollary 3.20. Gabriel Reconstruction Theorem

Suppose that X and Y are smooth projective varieties over k. If $\mathsf{Coh}(X) \simeq \mathsf{Coh}(Y)$, then $X \cong Y$.

Proof. See [Huyb Cor 5.23, 5.24]. ☐

4 The Derived Category $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(\mathbb{P}^n)$

In the section we focus on the structure of the derived category of coherent sheaves on $\mathbb{P}^n_k$.

Beĭlinson's Resolution of Diagonal

Consider the identity functor $\mathrm{id}\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$. From the previous section we note that this is isomorphic to the Fourier–Mukai functor $\Phi^{\mathcal{O}_\Delta}$. In the following we show that the diagonal sheaf $\mathcal{O}_\Delta$ has a finite resolution by vector bundles.

For $E, F \in \mathrm{Obj}(\mathcal{O}_{\mathbb{P}^n}\text{-}\mathsf{Mod})$, we define the **exterior tensor product** of E and F:

$$E \boxtimes F := p^* E \otimes q^* F \in \mathrm{Obj}(\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}\text{-}\mathsf{Mod}),$$

where $p, q\colon \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projections.

Theorem 4.1. Beĭlinson Resolution

Let L be the vector bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n \times \mathbb{P}^n$. The diagonal sheaf $\mathcal{O}_\Delta$ admits a resolution by vector bundles:

$$0 \longrightarrow \bigwedge^n L \longrightarrow \cdots \longrightarrow \bigwedge^2 L \longrightarrow L \longrightarrow \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

Proof. Consider the Euler exact sequence twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$:

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

Pulling back by p and q respectively, we obtain a morphism by the following composition:

$$q^*\Omega_{\mathbb{P}^n}(1) \longrightarrow q^*\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \cong \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}^{\oplus(n+1)} \cong p^*\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \longrightarrow p^*\mathcal{O}_{\mathbb{P}^n}(1).$$

Then tensoring $p^*\mathcal{O}_{\mathbb{P}^n}(-1)$, we obtain a morphism $\varepsilon\colon \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}$.

Geometrically, we consider $\mathbb{P}^n$ as the projectivisation of the $(n+1)$-dimensional vector space V. $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the **tautological bundle** of $\mathbb{P}^n$, whose fibre at $\ell \in \mathbb{P}^n$ is the line $\ell \leqslant V$ itself. $\Omega_{\mathbb{P}^n}(1)$ is dual to the tangent bundle $\mathcal{T}$ twisted by $\mathcal{O}_{\mathbb{P}^n}(-1)$. The fibre of $\Omega_{\mathbb{P}^n}(1)$ at $\ell \in \mathbb{P}^n$ is the annihilator of ℓ in $V^\vee$. The morphism ε is in fact the evaluation map $\varepsilon_{(\ell_1,\ell_2)}(v\otimes\varphi) = \varphi(v)$, where $v \in \ell_1$ and $\varphi \in \ell_2^\circ$.

Note that $\varepsilon_{(\ell_1,\ell_2)}$ is not surjective if and only if $\ell_1 = \ell_2$. It could be checked locally that the image of ε is the ideal sheaf of the diagonal $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$. Hence $\operatorname{coker}\varepsilon = \mathcal{O}_\Delta$. We have an exact sequence

$$\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega_{\mathbb{P}^n}(1) \xrightarrow{\ \varepsilon\ } \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

Then we can take the **Koszul resolution**:

$$0 \longrightarrow \bigwedge^n L \longrightarrow \cdots \longrightarrow \bigwedge^2 L \longrightarrow L \longrightarrow \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

where the morphism $\bigwedge^k L \to \bigwedge^{k-1} L$ is given by

$$s_1 \wedge \cdots \wedge s_k \longmapsto \sum_{j=1}^{p} (-1)^{j-1}\varepsilon(s_j) s_1 \wedge \cdots \wedge \widehat{s}_j \wedge \cdots \wedge s_k.$$ □

Let $L^{-k} := \bigwedge^k L \cong \mathcal{O}(-k) \boxtimes \Omega^k(k)$, $L^0 := \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}$, and $L^k = 0$ for $k > 0$. Then the theorem states that $L^\bullet$ is (quasi-)isomorphic to $\mathcal{O}_\Delta$ in the derived category $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n \times \mathbb{P}^n)$.

Corollary 4.2

$\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$ is generated as a triangulated category by the line bundles $\mathcal{O}_{\mathbb{P}^n}(-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}$.

Proof. Note that $\mathcal{O}_\Delta$ is the Fourier–Mukai kernel of the identity functor. Beĭlinson resolution produces a "resolution"[12] of the identity functor by Fourier–Mukai functors

$$\begin{aligned}\Phi^{L^{-k}} = \mathsf{R}p_*(p^*\mathcal{O}(-k) \otimes q^*\Omega^k(k) \otimes^{\mathsf{L}} q^*(-)) &\cong \mathcal{O}(-k) \otimes^{\mathsf{L}} \mathsf{R}p_*(\mathsf{L}q^*(\Omega^k(k) \otimes^{\mathsf{L}} -)) \\ &\cong \mathcal{O}(-k) \otimes^{\mathsf{L}}_k \mathsf{R}\Gamma(\mathbb{P}^n, \Omega^k(k) \otimes^{\mathsf{L}} -).\end{aligned}$$

[12] In fact this is the Beĭlinson spectral sequence. We choose not to go into this topic.

More specially, we split the Beĭlinson resolution into short exact sequences:

$$\begin{array}{c} 0 \longrightarrow L^{-n} \longrightarrow L^{-n+1} \longrightarrow M_{n-1} \longrightarrow 0 \\ 0 \longrightarrow M_{n-1} \longrightarrow L^{-n+2} \longrightarrow M_{n-2} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow M_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0 \end{array}$$

They are distinguished triangles in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n \times \mathbb{P}^n)$. For $F \in \mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n \times \mathbb{P}^n))$, applying the exact functor $\mathsf{R}q_*(\mathsf{L}p^*F \otimes^{\mathsf{L}} -)$ we obtain distinguished triangles

$$\Phi^{M_{k+1}}(F) \longrightarrow \Phi^{L^{-k}}(F) \longrightarrow \Phi^{M_k}(F) \xrightarrow{+1}$$

Note that $\Phi^{L^{-k}}(F) \cong \mathcal{O}(-k) \otimes^{\mathsf{L}}_k \mathsf{R}\Gamma(\mathbb{P}^n, \Omega^k(k) \otimes^{\mathsf{L}} F)$ is a tensor product of $\mathcal{O}_{\mathbb{P}^n}(-k)$ with a complex of finite-dimensional k-vector spaces. So $\Phi^{L^{-k}}(F)$ is contained in the triangulated subcategory generated by $\mathcal{O}_{\mathbb{P}^n}(-k)$. By induction, we have $\Phi^{M_k}(F) \in \langle \mathcal{O}_{\mathbb{P}^n}(-n), ..., \mathcal{O}_{\mathbb{P}^n}(-k)\rangle$. Finally, we have

$$F = \Phi^{\mathcal{O}_\Delta}(F) \in \langle \mathcal{O}_{\mathbb{P}^n}(-n), ..., \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}\rangle .$$

□

Remark. Note that tensoring the twisting sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ is an exact autoequivalence of $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$. Therefore $\mathcal{O}_{\mathbb{P}^n}(a-n), ..., \mathcal{O}_{\mathbb{P}^n}(a)$ also generate $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$ for any $a \in \mathbb{Z}$.

Remark. If we exchange the projections p and q, we obtain instead that

$$\Phi^{L^{-k}}(F) \cong \Omega^k(k) \otimes^{\mathsf{L}}_k \mathsf{R}\Gamma(\mathbb{P}^n, \mathcal{O}(-k) \otimes^{\mathsf{L}} F).$$

Using the same method we can show that $\mathcal{O}_{\mathbb{P}^n}, \Omega^1_{\mathbb{P}^n}(1), ..., \Omega^n_{\mathbb{P}^n}(n)$ also generate $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$.

Exceptional Sequence

Definition 4.3. Let $\mathcal{K}$ be a k-linear triangulated category. The objects $A_1, ..., A_n \in \mathrm{Obj}(\mathcal{K})$ form an **exceptional sequence**, if

$$\mathrm{Hom}_{\mathcal{K}}(A_i, A_j[n]) = \begin{cases} k, & \text{if } i = j,\ n = 0 \\ 0, & \text{if } i > j \text{ or if } i = j,\ n \neq 0 \end{cases}$$

If in addition $\mathrm{Hom}_{\mathcal{K}}(A_i, A_j[n]) = 0$ for all i, j and $n \neq 0$, then $A_1, ..., A_n \in \mathrm{Obj}(\mathcal{K})$ form an **strong exceptional sequence**.

If $A_1, ..., A_n$ generate $\mathcal{K}$ (i.e. the smallest triangulated subcategory of $\mathcal{K}$ containing $A_1, ..., A_n$ is $\mathcal{K}$ itself), then they form a **full exceptional sequence**.

Corollary 4.4

$\mathcal{O}_{\mathbb{P}^n}(-n), ..., \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}$ is a full strong exceptional sequence of $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$.

Proof. Using Beĭlinson resolution we have shown that they generate $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathbb{P}^n)$. That they form a strong exceptional sequence is due to the following facts (*cf.* [HartsAG II.5.13, III.5.1]):

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(i)[\ell]) = \mathrm{H}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \begin{cases} k, & \ell = 0; \\ 0, & \text{otherwise.} \end{cases}$$

For $i > j$,

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(j)[\ell]) = \mathrm{H}^\ell(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j-i)) = 0.$$

□

Tilting Object

Definition 4.5. Let $\mathcal{K}$ be a k-linear triangulated category. An object $T \in \mathrm{Obj}(\mathcal{K})$ is **tilting**, if:

1. $R := \mathrm{Hom}_{\mathcal{K}}(T, T)$ is a k-algebra of finite global dimension;
2. $\mathrm{Hom}_{\mathcal{K}}(T, T[i]) = 0$ for $i \neq 0$;
3. $\mathcal{K}$ is the smallest triangulated subcategory of $\mathcal{K}$ which contains T and is closed under isomorphisms and taking direct summands.[13]

Recall that the global dimension of R is the maximal projective dimension of an R-module.

Lemma 4.6

Let X be a smooth projective variety. If $E_1, \ldots, E_n$ is a full strong exceptional sequence in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$, then $E := \bigoplus_{i=1}^n E_i$ is a tilting object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$.

Proof. The proof of finite global dimension of $\mathrm{End}_{\mathcal{O}_X}(E)$ uses the path algebra of quiver. See [Craw Prop 6.6]. □

Example 4.7. We know that $\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ is a tilting sheaf on $\mathbb{P}^n$. Its endomorphism algebra is

$$R = \mathrm{Sym}^\bullet(V)/\,\mathrm{Sym}^{n+1}(V).$$

The non-vanishing Hom is given by

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(j)) \cong \mathrm{Sym}^{j-i}(V), \qquad i \leqslant j.$$

Theorem 4.8. Baer–Bondal Theorem

Let X be a smooth projective variety, and T be a tilting object in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$. Let $R := \mathrm{End}_{\mathcal{O}_X}(T)$ be the endomorphism algebra of T. Then the functor

$$\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(T, -) \colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \longrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$$

is an equivalence with quasi-inverse $- \otimes^{\mathsf{L}}_R T$. Here $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$ is the bounded derived category of finitely generated right R-modules.

Sketch of proof. We would like to show that $\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(T, -\otimes^{\mathsf{L}}_R T)$ is the identity functor on $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$.

[13] For unknown reason we say that T *classically generates* $\mathcal{K}$.

Observe that

$$\mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(T, R \otimes_R^{\mathsf{L}} T) = \mathsf{R}\operatorname{Hom}_{\mathcal{O}_X}(T, T) = \operatorname{Hom}_{\mathcal{O}_X}(T, T) = R,$$

since the non-zero Ext groups vanish. The smallest triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$ which contains R and its direct summands contains all finitely generated projective R-modules. Since R has finite global dimension, every finitely generated R-module admits a finite projective resolution. Hence the smallest triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$ which contains R and its direct summands is $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$ itself. This proves the claim.

Now $- \otimes_R^{\mathsf{L}} T$ identifies $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}^{\mathrm{fg}}\text{-}R)$ with the triangulated subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ classically generated by $R \otimes_R^{\mathsf{L}} T = T$. By definition, this subcategory is $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ itself. □

Quiver Representations

Definition 4.9. A **quiver** Q is a directed graph (Q_0, Q_1, s, t), where Q_0 is the set of vertices, Q_1 is the set of arrows, $s, t \colon Q_1 \to Q_0$ are the source and the target of an arrow respectively. A relation in Q with coefficient in k is a k-linear combination of paths of length at least 2, each with the same source and target. A **bound quiver** (Q, R) is a quiver Q with a set of relations R.

Definition 4.10. A **representation** V of the bound quiver (Q, R) consists of the following data:

- For each $i \in Q_0$, a k-vector space V_i;
- For each $a \in Q_i$, a k-linear map $\varphi_a \colon V_{s(i)} \to V_{t(i)}$;
- For each relation $r \in R$, the corresponding linear map is the zero map.

A morphism $\sigma \colon V \to W$ between representations of (Q, R) is the set of linear maps $\sigma_i \colon V_i \to W_i$ for each $i \in Q_0$ such that for each $a \in Q_1$, the following diagram commutes:

$$\begin{array}{ccc} V_{s(a)} & \xrightarrow{\varphi_a} & V_{t(a)} \\ {\scriptstyle \sigma_{s(a)}}\downarrow & & \downarrow{\scriptstyle \sigma_{t(a)}} \\ W_{s(a)} & \xrightarrow[\psi_a]{} & W_{t(a)} \end{array}$$

Definition 4.11. For a quiver Q, a path is the concatenation of some arrows in Q_1 (where a path of length 0 is an element of Q_0). The **path algebra** kQ is the free k-vector space generated by the paths in Q, with the multiplication

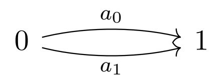
$$a \cdot b = \begin{cases} ab, & \text{if } s(a) = t(b); \\ 0, & \text{otherwise.} \end{cases}$$

The path algebra of a bound quiver (Q, R) is the quotient algebra $kQ/\langle R \rangle$.

Lemma 4.12

The category of finite-dimensional representations $\mathsf{Rep}_k(Q, R)$ of the bound quiver (Q, R) is equivalent to the category of finitely generated right $kQ/\langle R \rangle$-modules $\mathsf{Mod}^{\mathrm{fg}}\text{-}kQ/\langle R \rangle$.

Example 4.13. For $n = 1$, the Kronecker quiver Q is the quiver



without relations. The path algebra kQ is isomorphic to the endomorphism algebra $\operatorname{End}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. To see this, we write $\mathbb{P}^1_k = \mathbb{P}(ke_0 \oplus ke_1)$. Note that

$$\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)) = k, \quad \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)) \cong ke_0 \oplus ke_1, \quad \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}) = 0.$$

Putting $\mathcal{O}_{\mathbb{P}^1}$ at 0, $\mathcal{O}_{\mathbb{P}^1}(1)$ at 1, and e_i at the arrow a_i, we realise the endomorphism algebra of $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ as the path algebra of the Kronecker quiver.

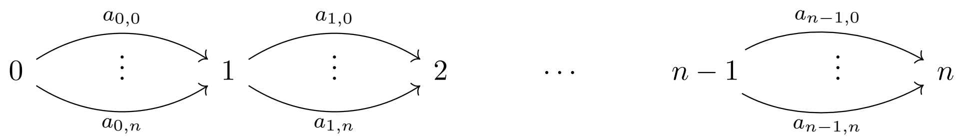
Corollary 4.14

There is a (bounded) derived equivalence between the category of coherent sheaves on $\mathbb{CP}^1$ and the category of finite-dimensional complex representations of the Kronecker quiver Q:

$$\mathsf{R}\operatorname{Hom}_{\mathcal{O}_{\mathbb{CP}^1}}(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(1), -)\colon \quad \mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(\mathbb{CP}^1) \longrightarrow \mathsf{D}^{\mathsf{b}}\, \mathsf{Rep}_{\mathbb{C}}\, Q.$$

This is the B-side of the homological mirror symmetry of $\mathbb{CP}^1$. For the A-side on the mirror of $\mathbb{CP}^1$ (which is the Landau–Ginzburg model on $\mathbb{C}^\times$), we need a lot more from sympletic geometry.[14]

Example 4.15. For $n \geqslant 2$, we define the Beĭlinson quiver Q of $\mathbb{P}^n$ to be



with the relations

$$R := \{a_{i,j}a_{i+1,\ell} - a_{i,\ell}a_{i+1,j} \colon 0 \leqslant j < \ell \leqslant n,\ 0 \leqslant i \leqslant n-1\}.$$

The path algebra $kQ/\langle R\rangle$ is isomorphic to the endomorphism algebra $\operatorname{End}_{\mathcal{O}_{\mathbb{P}^n}}(\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i))$.

Semi-Orthogonal Decomposition

The existence of a full exceptional sequence may be a too restrictive condition. Instead we may consider a weaker notion.

Definition 4.16. Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{K}$. We define the following full subcategories of $\mathcal{K}$:

- **Left orthogonal** ${}^\perp\mathcal{A} = \{T \in \operatorname{Obj}(\mathcal{K}) \colon \forall\, A \in \operatorname{Obj}(\mathcal{A})\ \operatorname{Hom}_{\mathcal{K}}(T, A) = 0\}$;
- **Right orthogonal** $\mathcal{A}^\perp = \{T \in \operatorname{Obj}(\mathcal{K}) \colon \forall\, A \in \operatorname{Obj}(\mathcal{A})\ \operatorname{Hom}_{\mathcal{K}}(A, T) = 0\}$.

Both ${}^\perp\mathcal{A}$ and $\mathcal{A}^\perp$ are triangulated.

Definition 4.17. A **semi-orthogonal decomposition** of a triangulated category $\mathcal{K}$ is a sequence $\mathcal{A}_1, ..., \mathcal{A}_n$ of full triangulated subcategories of $\mathcal{K}$ such that $\mathcal{A}_j \subseteq \mathcal{A}_i^\perp$ for $j < i$ and that $\mathcal{K}$ is generated by $\mathcal{A}_1, ..., \mathcal{A}_n$. We write $\mathcal{K} = \langle \mathcal{A}_1, ..., \mathcal{A}_n\rangle$.

[14] May be the minimal knowledge of Floer homology and Fukaya categories...See Ballard's *Meet Homological Mirror Symmetry* for a comprehensive treatment of $\mathbb{CP}^1$ (and T^2!)

Remark. If $E_1, ..., E_n$ is an exceptional sequence in $\mathcal{K}$, then $\mathcal{K}$ admits a semi-orthogonal decomposition

$$\mathcal{K} = \left\langle \langle E_1, ..., E_n \rangle^{\perp}, \langle E_1 \rangle, ..., \langle E_n \rangle \right\rangle.$$

Definition 4.18. A full triangulated subcategory $\mathcal{A}$ of $\mathcal{K}$ is called **admissible** if the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{K}$ admits both left and right adjoint.

In this case $\mathcal{K}$ admits semi-orthogonal decompositions $\mathcal{K} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$.

Corollary 4.19

Let X, Y be smooth projective varieties and $F \colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ be a fully faithful functor. Then

$$\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y) \cong \left\langle \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X), {}^{\perp}\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \right\rangle \cong \left\langle \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)^{\perp}, \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \right\rangle.$$

Proof. Identify $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ as the essential image under F in $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$. Orlov's result implies that F is an integral transform, and hence admits both left and right adjoint. □